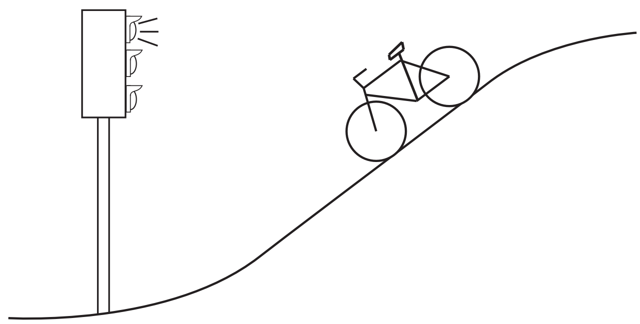
Optimal Braking Policy for Downhill Bicycling to Stoplight



When approaching a stoplight currently red, a bicyclist must often reduce their speed to avoid entering the intersection before their turn. When the bike is atop a hill, this is disadvantageous; hard-won potential energy is being wasted as brake heat. In practice, I have observed the following strategies being used:

1. Coast up to very last moment, then brake agressively

2. Come to a full stop at top of hill, wait until light is green then coast through

3. Brake continuously, though moderately, until approaching the intersection or the light turns green

To evaluate these braking policies, let's formalize the problem with control theory. The real-world objective is taken as the safe and timely arrival at a destination which is far beyond the stoplight (if the destination is at the stoplight, strategy (1) should be employed). The objective will then be taken to optimize velocity at the time the signal turns green. Note the following nomenclature:

Symbol	Quantity
$v(t)$	Velocity at time t
v_0	$v(t=0)$
$x(t)$	Position at time t, $x(t=0)=0$
d_0	Initial distance-to-travel to stop light
α	Acceleration induced by gravity
$b(t) > 0$	Braking force induced by bicyclist at time t
t_s	Time the stoplight switches green

The constraint induced by following traffic laws:

$$\begin{aligned} x(t_s) &\leq d_0 \\ &\Rightarrow \int_0^{t_s} v(t_s) \leq d_0 \end{aligned}$$

The dynamics for the system are (neglecting aerodynamic forces):

$$\begin{aligned}\frac{d}{dt}v\left(t\right) &= \alpha - b\left(t\right)\\ v\left(t\right) &= \alpha t + v_0 - \int_0^t b\left(\tau\right)d\tau\end{aligned}$$

Thus the constraint can be rewritten:

$$-\int_0^{t_s}\int_0^t b\left(\tau\right)d\tau dt \leq d_0 - v_0 t_s - \alpha\frac{t_s^2}{2}$$

Taking the case that $d_0 - v_0 t_s - \alpha\frac{t_s^2}{2} < 0$ (otherwise no braking effort will be needed), let $\underline{b} = d_0 - v_0 t_s - \alpha\frac{t_s^2}{2}$

$$\int_0^{t_s}\int_0^t b\left(\tau\right)d\tau dt \geq \underline{b}$$

And the objective can be re-written, with the expression for velocity above, and disregarding terms in the objective which are not affected by $b\left(t\right)$

$$\begin{aligned}\max_{b(t)} v\left(t_s\right) &= \max_{b(t)} \alpha t_s + v_0 - \int_0^{t_s} b\left(\tau\right)d\tau\\ &= \max_{b(t)} - \int_0^{t_s} b\left(\tau\right)d\tau\end{aligned}$$

Thus the optimization problem can be written as:

$$\begin{aligned}\min_{b(t)} \quad & \int_0^{t_s} b\left(\tau\right)d\tau\\ \text{such that} \quad & \int_0^{t_s}\int_0^t b\left(\tau\right)d\tau dt \geq \underline{b}\end{aligned}$$

So we want to minimize the first integral of $b\left(\tau\right)$, while ensuring the second intergral exceeds some minimum threshold $\underline{b}$. Intuitively, for an equivalent area under $b\left(\tau\right)$, having larger values earlier would increase the second integral. Thus, optimal braking policy is to brake quickly and early to a velocity from which simple coasting will satisfy the traffic law constraint.